Multivariate Difference-Differential Dimension Polynomials and New Invariants of Difference-Differential Field Extensions

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Abstract

In this paper we introduce a method of characteristic sets with respect to several term orderings for difference-differential polynomials. Using this technique, we obtain a method of computation of multivariate dimension polynomials of finitely generated difference-differential field extensions. Furthermore, we find new invariants of such extensions and show how the computation of multivariate difference-differential polynomials is applied to the equivalence problem for systems of algebraic difference-differential equations.

Keywords: Difference-differential field, dimension polynomial, reduction, characteristic set.

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1 Introduction

The role of Hilbert polynomials in commutative and homological algebra as well as in algebraic geometry and combinatorics is well known. A similar role in differential algebra is played by differential dimension polynomials, which describe in exact terms the freedom degree of a dynamic system, as well as the number of arbitrary constants in the general solution of a system of partial algebraic differential equations. The notion of a differential dimension polynomial was introduced by E. Kolchin [6] who proved the following fundamental result.

Theorem 1.1. *Let K be a differential field of zero characteristic with basic derivations $\delta_1, \dots, \delta_m$. Let Θ denote the free commutative semigroup generated by $\delta_1, \dots, \delta_m$, and for any $r \in \mathbf{N}$, let $\Theta(r) = \{\theta = \delta_1^{k_1} \dots \delta_m^{k_m} \in \Theta \mid \sum_{i=1}^{m} k_i \leq r\}$. Furthermore, let $L = K\langle \eta_1, \dots, \eta_n \rangle$ be a differential field extension of K generated by a finite set $\eta = \{\eta_1, \dots, \eta_n\}$. Then there exists a polynomial $\omega_{\eta|K}(t) \in \mathbf{Q}[t]$ such that $\omega_{\eta|K}(r) = trdeg_K K(\{\theta\eta_j | \theta \in \Theta(r), 1 \leq j \leq n\})$ for all sufficiently large $r \in \mathbf{Z}$. The degree of this polynomial does not exceed m and the numbers $d = \deg \omega_{\eta|K}$, a_m and a_d do not depend on the choice of the system of differential generators η of the extension L/K. Moreover, a_m is equal to the differential transcendence degree of L over K, that is, to the maximal number of elements $\xi_1, \dots, \xi_k \in L$ such that the set $\{\theta\xi_i | \theta \in \Theta, 1 \leq i \leq k\}$ is algebraically independent over K.*

The polynomial $\omega_{\eta|K}(t)$ is called the *differential dimension polynomial* of the extension L/K associated with the set of differential generators η.

If P is a prime differential ideal of a finitely generated differential algebra $R = K\{\zeta, \dots, \zeta_n\}$ over a differential field K, then the quotient field of R/P is a differential field extension of K generated by the images of ζ_i $(1 \leq i \leq n)$ in R/P. The corresponding differential dimension polynomial, therefore, characterizes the ideal P; it is denoted by $\omega_P(t)$. Assigning such polynomials to prime differential ideals has led to a number of new results on the Krull-type dimension of differential algebras and dimension of differential varieties (see, for example, [3], [4] and [5]). Furthermore, as it was shown by A. Mikhalev and E. Pankratev [13], one can naturally assign a differential dimension polynomial to a system of algebraic differential equations and this polynomial expresses the A. Einstein's strength of the system (see [1]). Methods of computation of (univariate) differential dimension polynomials and the strength of systems of differential equations via the Ritt-Kolchin technique of characteristic sets can be found, for example, in [14] and [8, Chapters 5, 9]. Note also, that there are quite many works on computation of dimension polynomials of differential, difference and difference-differential modules with the use of various generalizations of the Gröbner basis method (see, for example, [8, Chapters V - XI], [9], [10], [11], [12, Chapter 3], and [15]).

In this paper we develop a method of characteristic sets with respect to several orderings for algebras of difference-differential polynomials over a difference-differential fields whose basic set of derivations is partied into several disjoint subsets. We apply this method to prove the existence, outline a method of computation, and determine invariants of a multivariate dimension polynomial associated with a finite system of generators of a difference-differential field extension (and a partition of the basic sets of derivations). We also show that most of these invariants are not carried by univariate dimension polynomials and show how the consideration of the new invariants can be applied to the isomorphism problem for difference-differential field extensions and equivalence problem for systems of algebraic difference-differential equations.

2 Preliminaries

Throughout the paper, $\mathbf{N}, \mathbf{Z}$, $\mathbf{Q}$, and $\mathbf{R}$ denote the sets of all non-negative integers, integers, rational numbers, and real numbers, respectively. $\mathbf{Q}[t]$ will denote the ring of polynomials in one variable t with rational coefficients.

By a *difference-differential ring* we mean a commutative ring R together with finite sets $\Delta = \{\delta_1, \dots, \delta_m\}$ and $\sigma = \{\alpha_1, \dots, \alpha_n\}$ of derivations and automorphisms of R, respectively, such that any two mappings of the set $\Delta \bigcup \sigma$ commute. The set $\Delta \bigcup \sigma$ is called the *basic set* of the difference-differential ring R, which is also called a Δ-σ-ring. If R is a field, it is called a *difference-differential field* or a Δ-σ-field. Furthermore, in what follows, we denote the set $\{\alpha_1, \dots, \alpha_n, \alpha_1^{-1}, \dots, \alpha_n^{-1}\}$ by σ^*.

If R is a difference-differential ring with a basic set $\Delta \bigcup \sigma$ described above,

then Λ will denote the free commutative semigroup of all power products of the form $\lambda = \delta_1^{k_1} \dots \delta_m^{k_m} \alpha_1^{l_1} \dots \alpha_n^{l_n}$ where $k_i \in \mathbf{N}$, $l_j \in \mathbf{Z}$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$. For any such an element λ, we set $\lambda_\Delta = \delta_1^{k_1} \dots \delta_m^{k_m}$, $\lambda_\sigma = \alpha_1^{l_1} \dots \alpha_n^{l_n}$, and denote by Λ_Δ and Λ_σ the commutative semigroup of power products $\delta_1^{k_1} \dots \delta_m^{k_m}$ and the commutative group of elements of the form $\alpha_1^{l_1} \dots \alpha_n^{l_n}$, respectively. The *order* of λ is defined as $ord\,\lambda = \sum_{i=1}^m k_i + \sum_{j=1}^n |l_j|$, and for every $r \in \mathbf{N}$, we set $\Lambda(r) = \{\lambda \in \Lambda \mid ord\,\lambda \leq r\}$ $(r \in \mathbf{N})$.

A subring (ideal) R_0 of a Δ-σ-ring R is said to be a difference-differential (or Δ-σ-) subring of R (respectively, difference-differential (or Δ-σ-) ideal of R) if R_0 is closed with respect to the action of any operator of $\Delta \bigcup \sigma^*$. If a prime ideal P of R is closed with respect to the action of $\Delta \bigcup \sigma^*$, it is called a *prime* difference-differential (or Δ-σ-) *ideal* of R.

If R is a Δ-σ-field and R_0 a subfield of R which is also a Δ-σ-subring of R, then R_0 is said to be a Δ-σ-subfield of R; R, in turn, is called a difference-differential (or Δ-σ-) field extension or a Δ-σ-overfield of R_0. In this case we also say that we have a Δ-σ-field extension R/R_0.

If R is a Δ-σ-ring and $\Sigma \subseteq R$, then the intersection of all Δ-ideals of R containing the set Σ is, obviously, the smallest Δ-σ-ideal of R containing Σ. This ideal is denoted by $[\Sigma]$. (It is clear that $[\Sigma]$ is generated, as an ideal, by the set $\{\lambda\xi | \xi \in \Sigma,\ \lambda \in \Lambda\}$). If the set Σ is finite, $\Sigma = \{\xi_1, \dots, \xi_q\}$, we say that the Δ-ideal $I = [\Sigma]$ is finitely generated (we write this as $I = [\xi_1, \dots, \xi_q]$) and call $\xi_1, \dots, \xi_q$ difference-differential (or Δ-σ-) generators of I.

If K_0 is a Δ-σ-subfield of the Δ-σ-field K and $\Sigma \subseteq K$, then the intersection of all Δ-σ-subfields of K containing K_0 and Σ is the unique Δ-σ-subfield of K containing K_0 and Σ and contained in every Δ-σ-subfield of K containing K_0 and Σ. It is denoted by $K_0\langle\Sigma\rangle$. If $K = K_0\langle\Sigma\rangle$ and the set Σ is finite, $\Sigma = \{\eta_1, \dots, \eta_s\}$, then K is said to be a finitely generated Δ-σ-extension of K_0 with the set of Δ-σ-generators $\{\eta_1, \dots, \eta_s\}$. In this case we write $K = K_0\langle\eta_1, \dots, \eta_s\rangle$. It is easy to see that the field $K_0\langle\eta_1, \dots, \eta_s\rangle$ coincides with the field $K_0(\{\lambda\eta_i | \lambda \in \Lambda, 1 \leq i \leq s\})$.

Let R and S be two difference-differential rings with the same basic set $\Delta \bigcup \sigma$, so that elements of the sets Δ and σ act on each of the rings as mutually commuting derivations and automorphisms, respectively. A ring homomorphism $\phi : R \longrightarrow S$ is called a *difference-differential* (or Δ-σ-) *homomorphism* if $\phi(\tau a) = \tau\phi(a)$ for any $\tau \in \Delta \bigcup \sigma$, $a \in R$.

If K is a difference-differential (Δ-σ-) field and $Y = \{y_1, \dots, y_s\}$ is a finite set of symbols, then one can consider the countable set of symbols $\Lambda Y = \{\lambda y_j | \lambda \in \Lambda, 1 \leq j \leq s\}$ and the polynomial ring $R = K[\{\lambda y_j | \lambda \in \Lambda, 1 \leq j \leq s\}]$ in the set of indeterminates ΛY over the field K. This polynomial ring is naturally viewed as a Δ-σ-ring where $\tau(\lambda y_j) = (\tau\lambda) y_j$ for any $\tau \in \Delta \bigcup \sigma$, $\lambda \in \Lambda$, $1 \leq j \leq s$, and the elements of $\Delta \bigcup \sigma$ act on the coefficients of the polynomials of R as they act in the field K. The ring R is called a *ring of difference-differential* (or Δ-σ-) *polynomials* in the set of differential (Δ-σ-)indeterminates $y_1, \dots, y_s$ over K. This ring is denoted by $K\{y_1, \dots, y_s\}$ and its elements are called difference-differential (or Δ-σ-) polynomials.

Let $L = K\langle \eta_1, \dots, \eta_s \rangle$ be a difference-differential field extension of K generated by a finite set $\eta = \{\eta_1, \dots, \eta_s\}$. As a field, $L = K(\{\lambda\eta_j | \lambda \in \Lambda, 1 \le j \le s\})$.

The following is a unified version of E. Kolchin's theorem on differential dimension polynomial and the author's theorem on the dimension polynomial of a difference field extension (see [9] or [12, Theorem 4.2.5]).

Theorem 2.1. *With the above notation, there exists a polynomial $\phi_{\eta|K}(t) \in \mathbf{Q}[t]$ such that*

(i) $\phi_{\eta|K}(r) = trdeg_K K(\{\lambda\eta_j | \lambda \in \Lambda(r), 1 \le j \le s\})$ *for all sufficiently large* $r \in \mathbf{Z}$*;*

(ii) $\deg \phi_{\eta|K} \le m+n$ *and* $\phi_{\eta|K}(t)$ *can be written as* $\phi_{\eta|K}(t) = \sum_{i=0}^{m+n} a_i \binom{t+i}{i}$ *where* $a_0, \dots, a_{m+n} \in \mathbf{Z}$ *and* $2^n | a_{m+n}$ *.*

(iii) $d = \deg \phi_{\eta|K}$*,* a_{m+n} *and* a_d *do not depend on the set of difference-differential generators* η *of* L/K *(*$a_d \ne a_{m+n}$ *if and only if* $d < m+n$*). Moreover,* $\frac{a_{m+n}}{2^n}$ *is equal to the difference-differential transcendence degree of* L *over* K *(denoted by* Δ*-*σ*-*$trdeg_K L$*), that is, to the maximal number of elements* $\xi_1, \dots, \xi_k \in L$ *such that the family* $\{\lambda\xi_i | \lambda \in \Lambda, 1 \le i \le k\}$ *is algebraically independent over* K*.*

The polynomial whose existence is established by this theorem is called a *univariate difference-differential* (or Δ-σ-) *dimension polynomial* of the extension L/K associated with the system of difference-differential generators η.

3 Partition of the basic set of derivations and the formulation of the main theorem

Let K be a difference-differential field of zero characteristic with basic sets $\Delta = \{\delta_1, \dots, \delta_m\}$ and $\sigma = \{\alpha_1, \dots, \alpha_n\}$ of derivations and automorphisms, respectively. Suppose that the set of derivations is represented as the union of p disjoint subsets ($p \ge 1$):

$$\Delta = \Delta_1 \bigcup \dots \bigcup \Delta_p \tag{3. 1}$$

$$\text{where } \Delta_1 = \{\delta_1, \dots, \delta_{m_1}\}, \Delta_2 = \{\delta_{m_1+1}, \dots, \delta_{m_1+m_2}\}, \dots,$$

$$\Delta_p = \{\delta_{m_1+\dots+m_{p-1}+1}, \dots, \delta_m\} \ \ (m_1 + \dots + m_p = m).$$

If $\lambda = \delta_1^{k_1} \dots \delta_m^{k_m} \alpha_1^{l_1} \dots \alpha_n^{l_n} \in \Lambda$ ($k_i \in \mathbf{N}$, $l_j \in \mathbf{Z}$), then the order of λ with respect to Δ_i ($1 \le i \le p$) is defined as $\sum_{\nu=m_1+\dots+m_{i-1}+1}^{m_1+\dots+m_i} k_\nu$; it is denoted by $ord_i\lambda$. (If $i = 1$, the last sum is replaced by $k_1 + \dots + k_{m_1}$.) The number $ord_\sigma\lambda = \sum_{j=1}^{n} |l_j|$ is called the order of λ with respect to σ. Furthermore, for any

$r_1, \dots, r_{p+1} \in \mathbf{N}$, we set

$$\Lambda(r_1, \dots, r_{p+1}) = \{\lambda \in \Lambda \,|\, ord_i \lambda \le r_i \ (i = 1, \dots, p) \text{ and } ord_\sigma \lambda \le r_{p+1}\}.$$

In what follows, for any permutation $(j_1, \dots, j_{p+1})$ of the set $\{1, \dots, p+1\}$, $<_{j_1,\dots,j_{p+1}}$ will denote the lexicographic order on $\mathbf{N}^{p+1}$ such that $(r_1, \dots, r_{p+1}) <_{j_1,\dots,j_{p+1}} (s_1, \dots, s_{p+1})$ if and only if either $r_{j_1} < s_{j_1}$ or there exists $k \in \mathbf{N}$, $1 \le k \le p$, such that $r_{j_\nu} = s_{j_\nu}$ for $\nu = 1, \dots, k$ and $r_{j_{k+1}} < s_{j_{k+1}}$.

Furthermore, if $\Sigma \subseteq \mathbf{N}^{p+1}$, then Σ' denotes the set $\{e \in \Sigma | e$ is a maximal element of Σ with respect to one of the $(p+1)!$ lexicographic orders $<_{j_1,\dots,j_{p+1}}\}$.

For example, if $\Sigma = \{(3,0,2), (2,1,1), (0,1,4), (1,0,3), (1,1,6), (3,1,0), (1,2,0)\} \subseteq \mathbf{N}^3$, then $\Sigma' = \{(3,0,2), (3,1,0), (1,1,6), (1,2,0)\}$.

Theorem 3.1. *Let $L = K\langle \eta_1, \dots, \eta_s \rangle$ be a Δ-σ-field extension generated by a set $\eta = \{\eta_1, \dots, \eta_s\}$. Then there exists a polynomial $\Phi_\eta(t_1, \dots, t_{p+1})$ in $(p+1)$ variables $t_1, \dots, t_{p+1}$ with rational coefficients such that*

(i) $\Phi_\eta(r_1, \dots, r_{p+q}) = trdeg_K K(\bigcup_{j=1}^{s} \Lambda(r_1, \dots, r_{p+1})\eta_j)$

for all sufficiently large $(r_1, \dots, r_{p+1}) \in \mathbf{N}^{p+1}$ (it means that there exist nonnegative integers $s_1, \dots, s_{p+1}$ such that the last equality holds for all $(r_1, \dots, r_{p+1}) \in \mathbf{N}^{p+1}$ with $r_1 \ge s_1, \dots, r_{p+1} \ge s_{p+1}$);

(ii) *$deg_{t_i} \Phi_\eta \le m_i$ $(1 \le i \le p)$ and $deg_{t_{p+1}} \Phi_\eta \le n$, so that $deg\, \Phi_\eta \le m + n$ and $\Phi_\eta(t_1, \dots, t_{p+1})$ can be represented as*

$$\Phi_\eta(t_1, \dots, t_{p+1}) = \sum_{i_1=0}^{m_1} \dots \sum_{i_p=0}^{m_p} \sum_{i_{p+1}=0}^{n} a_{i_1 \dots i_{p+1}} \binom{t_1 + i_1}{i_1} \dots \binom{t_{p+1} + i_{p+1}}{i_{p+1}}$$

where $a_{i_1 \dots i_{p+1}} \in \mathbf{Z}$ and $2^n \,|\, a_{m_1 \dots m_p n}$.

(iii) *Let $E_\eta = \{(i_1, \dots, i_{p+1}) \in \mathbf{N}^{p+1} \,|\, 0 \le i_k \le m_k$ for $k = 1, \dots, p$, $0 \le i_{p+1} \le n$, and $a_{i_1 \dots i_{p+1}} \ne 0\}$. Then $d = deg\, \Phi_\eta$, $a_{m_1 \dots m_{p+1}}$, elements $(k_1, \dots, k_{p+1}) \in E'_\eta$, the corresponding coefficients $a_{k_1 \dots k_{p+1}}$ and the coefficients of the terms of total degree d do not depend on the choice of the system of Δ-σ-generators η.*

Definition 3.2. *The polynomial $\Phi_\eta(t_1, \dots, t_{p+1})$ is said to be the difference-differential* (or Δ-σ-) *dimension polynomial of the Δ-σ-field extension L/K associated with the set of Δ-σ-generators η and partition* (3.1) *of the basic set of derivations.*

The Δ-σ-dimension polynomial associated with partition (3.1) has the following interpretation as the strength of a system of difference-differential equations.

Let us consider a system of partial difference-differential equations

$$A_i(f_1, \dots, f_s) = 0 \qquad (i = 1, \dots, q) \tag{3. 2}$$

over a field of functions of m real variables $x_1, \dots, x_m$ ($f_1, \dots, f_s$ are unknown functions of $x_1, \dots, x_m$). Suppose that $\Delta = \{\delta_1, \dots, \delta_m\}$ where δ_i is the partial differentiation $\partial/\partial x_i$ ($i = 1, \dots, m$) and the basic set of automorphisms $\sigma = \{\alpha_1, \dots, \alpha_m\}$ consists of m shifts of arguments, $f(x_1, \dots, x_m) \mapsto f(x_1, \dots, x_{i-1}, x_i + h_i, x_{i+1}, \dots, x_m)$ ($1 \le i \le m$, $h_1, \dots, h_m \in \mathbf{R}$). Thus, we assume that the left-hand sides of the equations in (3.2) contain unknown functions f_i, their partial derivatives, their images under the shifts α_j, and various compositions of such shifts and partial derivations. Furthermore, we suppose that system (3.2) is algebraic, that is, all $A_i(y_1, \dots, y_s)$ are elements of a ring of Δ-σ-polynomials $K\{y_1, \dots, y_s\}$ with coefficients in some functional Δ-σ-field K.

Let us consider a grid with equal cells of dimension $h_1 \times \cdots \times h_m$ that fills the whole space $\mathbf{R}^m$. We fix some node $\mathcal{P}$ and say that *a node $\mathcal{Q}$ has order i* if the shortest path from $\mathcal{P}$ to $\mathcal{Q}$ along the edges of the grid consists of i steps (by a step we mean a path from a node of the grid to a neighbor node along the edge between them). We also fix partition (3.1) of the set of basic derivations Δ (such a partition can be, for example, a natural separation of (all or some) derivations with respect to coordinates and the derivation with respect to time).

For any $r_1, \dots, r_{p+1} \in \mathbf{N}$, let us consider the values of the unknown functions $f_1, \dots, f_s$ and their partial derivatives, whose order with respect to Δ_i does not exceed r_i ($1 \le i \le p$), at the nodes whose order does not exceed r_{p+1}. If $f_1, \dots, f_s$ should not satisfy any system of equations (or any other condition), these values can be chosen arbitrarily. Because of the system (and equations obtained from the equations of the system by partial differentiations and transformations of the form $f_j(x_1, \dots, x_m) \mapsto f_j(x_1 + k_1h_1, \dots, x_m + k_mh_m)$ with $k_1, \dots, k_m \in \mathbf{Z}$, $1 \le j \le s$), the number of independent values of the functions $f_1, \dots, f_s$ and their partial derivatives whose ith order does not exceed r_i ($1 \le i \le p$) at the nodes of order $\le r_{p+1}$ decreases. This number, which is a function of $p+1$ variables $r_1, \dots, r_{p+1}$, is the “measure of strength” of the system in the sense of A. Einstein. We denote it by $S_{r_1,\dots,r_{p+1}}$.

Suppose that the Δ-σ-ideal J generated in the ring $K\{y_1, \dots, y_s\}$ by the Δ-σ-polynomials $A_1, \dots, A_q$ is prime (e. g., the polynomials are linear). Then the field of fractions L of the Δ-σ-integral domain $K\{y_1, \dots, y_s\}/J$ has a natural structure of a Δ-σ-field extension of K generated by the finite set $\eta = \{\eta_1, \dots, \eta_s\}$ where η_i is the canonical image of y_i in $K\{y_1, \dots, y_s\}/J$ ($1 \le i \le s$). It is easy to see that the Δ-σ-dimension polynomial $\Phi_\eta(t_1, \dots, t_{p+1})$ of the extension L/K associated with the system of Δ-σ-generators η has the property that $\Phi_\eta(r_1, \dots, r_{p+1}) = S_{r_1,\dots,r_{p+1}}$ for all sufficiently large $(r_1, \dots, r_{p+q}) \in \mathbf{N}^{p+1}$, so this dimension polynomial is the measure of strength of the system of difference-differential equations (3.2) in the sense of A. Einstein.

4 Numerical polynomials of subsets of $\mathbf{N}^m \times \mathbf{Z}^n$

Definition 4.1. *A polynomial $f(t_1, \dots, t_p)$ in p variables $t_1, \dots, t_p$ ($p \ge 1$) with rational coefficients is called* numerical *if $f(r_1, \dots, r_p) \in \mathbf{Z}$ for all sufficiently large $(r_1, \dots, r_p) \in \mathbf{Z}^p$.*

Of course, every polynomial with integer coefficients is numerical. As an example of a numerical polynomial in p variables with noninteger coefficients ($p \geq 1$) one can consider $\prod_{i=1}^{p} \binom{t_i}{m_i}$ where $m_1, \dots, m_p \in \mathbf{N}$. (As usual, $\binom{t}{k}$ ($k \geq 1$) denotes the polynomial $\frac{t(t-1)\dots(t-k+1)}{k!}$, $\binom{t}{0} = 1$, and $\binom{t}{k} = 0$ if $k < 0$.)

The following theorem proved in [8, Chapter 2] gives the "canonical" representation of a numerical polynomial in several variables.

Theorem 4.2. *Let $f(t_1, \dots, t_p)$ be a numerical polynomial in p variables and let $deg_{t_i} f = m_i$ ($m_1, \dots, m_p \in \mathbf{N}$). Then $f(t_1, \dots, t_p)$ can be represented as*

$$f(t_1, \dots t_p) = \sum_{i_1=0}^{m_1} \dots \sum_{i_p=0}^{m_p} a_{i_1 \dots i_p} \binom{t_1 + i_1}{i_1} \dots \binom{t_p + i_p}{i_p} \qquad (4.\ 1)$$

with uniquely defined integer coefficients $a_{i_1 \dots i_p}$.

In what follows, we deal with subsets of $\mathbf{N}^m \times \mathbf{Z}^n$ ($m, n \geq 1$) and a fixed partition of the set $\mathbf{N}_m = \{1, \dots, m\}$ into p disjoint subsets ($p \geq 1$):

$$\mathbf{N}_m = N_1 \bigcup \dots N_p \qquad (4.\ 2)$$

where $N_1 = \{1, \dots, m_1\}, \dots, N_p = \{m_1 + \dots + m_{p-1} + 1, \dots, m\}$ $(m_1 + \dots + m_p = m)$. If $a = (a_1, \dots, a_{m+n}) \in \mathbf{N}^m \times \mathbf{Z}^n$ we denote the numbers $\sum_{i=1}^{m_1} a_i$, $\sum_{i=m_1+1}^{m_1+m_2} a_i, \dots, \sum_{i=m_1+\dots+m_{p-1}+1}^{m} a_i$, $\sum_{i=m+1}^{m+n} |a_i|$ by $ord_1 a, \dots, ord_{p+1} a$, respectively. Furthermore, we consider the set $\mathbf{Z}^n$ as a union

$$\mathbf{Z}^n = \bigcup_{1 \leq j \leq 2^n} \mathbf{Z}_j^{(n)} \qquad (4.\ 3)$$

where $\mathbf{Z}_1^{(n)}, \dots, \mathbf{Z}_{2^n}^{(n)}$ are all different Cartesian products of n sets each of which is either $\mathbf{N}$ or $\mathbf{Z}_- = \{a \in \mathbf{Z} | a \leq 0\}$. We assume that $\mathbf{Z}_1^{(n)} = \mathbf{N}^n$ and call $\mathbf{Z}_j^{(n)}$ the *jth orthant* of the set $\mathbf{Z}^n$ ($1 \leq j \leq 2^n$). The set $\mathbf{N}^m \times \mathbf{Z}^n$ is considered as a partially ordered set with the order $\trianglelefteq$ such that $(e_1, \dots, e_m, f_1, \dots, f_n) \trianglelefteq (e'_1, \dots, e'_m, f'_1, \dots, f'_n)$ if and only if $(f_1, \dots, f_n)$ and $(f'_1, \dots, f'_n)$ belong to the same orthant $\mathbf{Z}_k^{(n)}$ and the $(m+n)$-tuple $(e_1, \dots, e_m, |f_1|, \dots, |f_n|)$ is less than $(e'_1, \dots, e'_m, |f'_1|, \dots, |f'_n|)$ with respect to the product order on $\mathbf{N}^{m+n}$.

In what follows, for any set $A \subseteq \mathbf{N}^m \times \mathbf{Z}^n$, W_A will denote the set of all elements of $\mathbf{N}^m \times \mathbf{Z}^n$ that do not exceed any element of A with respect to the order $\trianglelefteq$. (Thus, $w \in W_A$ if and only if there is no $a \in A$ such that $a \trianglelefteq w$.) Furthermore, for any $r_1, \dots r_{p+1} \in \mathbf{N}$, $A(r_1, \dots r_{p+1})$ will denote the set of all elements $x = (x_1, \dots, x_m, x'_1, \dots, x'_n) \in A$ such that $ord_i x \leq r_i$ ($i = 1, \dots, p+1$).

The above notation can be naturally restricted to subsets of $\mathbf{N}^m$. If $E \subseteq \mathbf{N}^m$ and $s_1, \dots, s_p \in \mathbf{N}$, then $E(s_1, \dots, s_p)$ will denote the set $\{e = (e_1, \dots, e_m) \in E \,|\, ord_i(e_1, \dots, e_m, 0, \dots, 0) \leq s_i$ for $i = 1, \dots, p\}$ ($(e_1, \dots, e_m, 0, \dots, 0)$ ends with n zeros; it is treated as a point in $\mathbf{N}^m \times \mathbf{Z}^n$.) Furthermore V_E will denote the set of all m-tuples $v = (v_1, \dots, v_m) \in \mathbf{N}$ which are not greater than or

equal to any m-tuple from E with respect to the product order on $\mathbf{N}^m$. (Recall that the product order on $\mathbf{N}^m$ is a partial order $\leq_P$ on $\mathbf{N}^m$ such that $c = (c_1, \dots, c_m) \leq_P c' = (c'_1, \dots, c'_m)$ if and only if $c_i \leq c'_i$ for all $i = 1, \dots, m$. If $c \leq_P c'$ and $c \neq c'$, we write $c <_P c'$). Clearly, $v = (v_1, \dots, v_m) \in V_E$ if and only if for any element $(e_1, \dots, e_m) \in E$, there exists $i \in \mathbf{N}, 1 \leq i \leq m$, such that $e_i > v_i$.

The following two theorems proved in [8, Chapter 2] generalize the well-known Kolchin's result on the numerical polynomials associated with subsets of $\mathbf{N}^m$ (see [7, Chapter 0, Lemma 16]) and give an explicit formula for the numerical polynomials in p variables associated with a finite subset of $\mathbf{N}^m$.

Theorem 4.3. *Let E be a subset of $\mathbf{N}^m$ where $m = m_1 + \dots + m_p$ for some non-negative integers $m_1, \dots, m_p$ ($p \geq 1$). Then there exists a numerical polynomial $\omega_E(t_1, \dots, t_p)$ with the following properties:*

(i) *$\omega_E(r_1, \dots, r_p) = Card\, V_E(r_1, \dots, r_p)$ for all sufficiently large $(r_1, \dots, r_p) \in$* $\mathbf{N}^p$. (As usual, $Card\, M$ denotes the number of elements of a finite set M).

(ii) *$deg_{t_i} \omega_E \leq m_i$ for all $i = 1, \dots, p$.*

(iii) *$deg\, \omega_E = m$ if and only if $E = \emptyset$. Then $\omega_E(t_1, \dots, t_p) = \prod_{i=1}^{p} \binom{t_i + m_i}{m_i}$.*

Definition 4.4. *The polynomial $\omega_E(t_1, \dots, t_p)$ is called the dimension polynomial of the set $E \subseteq \mathbf{N}^m$ associated with the partition $(m_1, \dots, m_p)$ of m.*

Theorem 4.5. *Let $E = \{e_1, \dots, e_q\}$ ($q \geq 1$) be a finite subset of $\mathbf{N}^m$ and let a partition (4.2) of the set $\mathbf{N}_m$ into p disjoint subsets $N_1, \dots, N_p$ be fixed. Let $e_i = (e_{i1}, \dots, e_{im})$ $(1 \leq i \leq q)$ and for any $l \in \mathbf{N}$, $0 \leq l \leq q$, let $\Gamma(l, q)$ denote the set of all l-element subsets of the set $\mathbf{N}_q = \{1, \dots, q\}$. Furthermore, for any $\sigma \in \Gamma(l, q)$, let $\bar{e}_{\emptyset j} = 0$, $\bar{e}_{\sigma j} = \max\{e_{ij} | i \in \sigma\}$ if $\sigma \neq \emptyset$ $(1 \leq j \leq m)$, and $b_{\sigma k} = \sum_{h \in N_k} \bar{e}_{\sigma h}$ $(k = 1, \dots, p)$. Then*

$$\omega_E(t_1, \dots, t_p) = \sum_{l=0}^{q} (-1)^l \sum_{\sigma \in \Gamma(l,\, q)} \prod_{j=1}^{p} \binom{t_j + m_j - b_{\sigma j}}{m_j} \tag{4. 4}$$

Remark. It is clear that if $E \subseteq \mathbf{N}^m$ and E^* is the set of all minimal elements of the set E with respect to the product order on $\mathbf{N}^m$, then the set E^* is finite and $\omega_E(t_1, \dots, t_p) = \omega_{E^*}(t_1, \dots, t_p)$. Thus, Theorem 4.5 gives an algorithm that allows one to find a numerical polynomial associated with any subset of $\mathbf{N}^m$ (and with a given partition of the set $\{1, \dots, m\}$): one should first find the set of all minimal points of the subset and then apply Theorem 4.5.

The following result can be obtained precisely in the same way as Theorem 3.4 of [10] (the only difference is that the proof in the mentioned paper uses Theorem 3.2 of [10] in the case $p = 2$, while the proof of the theorem below should refer to the Theorem 3.2 of [10] where p is any positive integer).

Theorem 4.6. *Let $A \subseteq \mathbf{N}^m \times \mathbf{Z}^n$ and let partition (4.2) of $\mathbf{N}_m$ be fixed. Then there exists a numerical polynomial $\phi_A(t_1, \dots, t_{p+1})$ in $p+1$ variables such that*

(i) $\phi_A(r_1,\dots,r_{p+1}) = Card\, W_A(r_1,\dots,r_{p+1})$ *for all sufficiently large* $(r_1,\dots,r_{p+1}) \in \mathbf{N}^{p+1}$.

(ii) $deg_{t_i}\phi_A \le m_i$ *(*$1 \le i \le p$*),* $deg_{t_{p+1}}\phi_A \le n$ *and the coefficient of* $t_1^{m_1}\dots t_p^{m_p}t_{p+1}^{n}$ *in* ϕ_A *is of the form* $\dfrac{2^n a}{m_1!\dots m_p!n!}$ *with* $a \in \mathbf{Z}$.

(iii) *Let us consider a mapping* $\rho : \mathbf{N}^m \times \mathbf{Z}^n \longrightarrow \mathbf{N}^{m+2n}$ *such that*

$\rho((e_1,\dots,e_{m+n}) = (e_1,\dots,e_m, \max\{e_{m+1},0\},\dots,\max\{e_{m+n},0\},$
$\max\{-e_{m+1},0\},\dots,\max\{-e_{m+n},0\})$.

Let $B = \rho(A)\bigcup\{\bar{e}_1,\dots,\bar{e}_n\}$ *where* $\bar{e}_i$ *(*$1 \le i \le n$*) is a* $(m+2n)$*-tuple in* $\mathbf{N}^{m+2n}$ *whose* $(m+i)$*th and* $(m+n+i)$*th coordinates are equal to 1 and all other coordinates are equal to 0. Then* $\phi_A(t_1,\dots,t_{p+1}) = \omega_B(t_1,\dots,t_{p+1})$ *where* $\omega_B(t_1,\dots,t_{p+1})$ *is the dimension polynomial of the set* B *(see* Definition 4.4*) associated with the partition* $\mathbf{N}_{m+2n} = \{1,\dots,m_1\}\bigcup\{m_1+1,\dots,m_1+m_2\}\bigcup\dots\bigcup\{m_1+\dots+m_{p-1}+1,\dots,m\}\bigcup\{m+1,\dots,m+2n\}$ *of the set* $\mathbf{N}_{m+2n}$.

(iv) *If* $A = \emptyset$*, then*

$$\phi_A(t_1,\dots,t_{p+1}) = \binom{t_1+m_1}{m_1}\dots\binom{t_p+m_p}{m_p}\sum_{i=0}^{n}(-1)^{n-i}2^i\binom{n}{i}\binom{t_{p+1}+i}{i}. \tag{4. 5}$$

The polynomial $\phi_A(t_1,\dots,t_{p+1})$ is called the *dimension polynomial* of the set $A \subseteq \mathbf{N}^m \times \mathbf{Z}^n$ associated with partition (4.2) of $\mathbf{N}_m$.

5 Proof of the main theorem and computation of difference-differential dimension polynomials via characteristic sets

In this section we prove Theorem 3.1 and give a method of computation of difference-differential dimension polynomials of Δ-σ-field extensions based on constructing a characteristic set of the defining prime Δ-σ-ideal of the extension.

In what follows we use the conventions of section 3. In particular, we assume that partition (3.1) of the set of basic derivations $\Delta = \{\delta_1,\dots,\delta_m\}$ is fixed.

Let us consider $p+1$ total orderings $<_1,\dots,<_p,<_\sigma$ of the set of power products Λ such that

$\lambda = \delta_1^{k_1}\dots\delta_m^{k_m}\alpha_1^{l_1}\dots\alpha_n^{l_n} <_i \lambda' = \delta_1^{k'_1}\dots\delta_m^{k'_m}\alpha_1^{l'_1}\dots\alpha_n^{l'_n}$ $(1 \le i \le p)$ if and only if $(ord_i\lambda, ord\,\lambda, ord_1\lambda,\dots,ord_{i-1}\lambda, ord_{i+1}\lambda,\dots,ord_p\lambda, ord_\sigma\lambda, k_{m_1+\dots+m_{i-1}+1},\dots,$ $k_{m_1+\dots+m_i}, k_1,\dots,k_{m_1+\dots+m_{i-1}}, k_{m_1+\dots+m_i+1},\dots,k_m, |l_1|,\dots,|l_n|, l_1,\dots,l_n)$ is less than $(ord_i\lambda', ord\,\lambda', ord_1\lambda',\dots,ord_{i-1}\lambda', ord_{i+1}\lambda',\dots,ord_p\lambda', ord_\sigma\lambda',$ $k'_{m_1+\dots+m_{i-1}+1},\dots,k'_{m_1+\dots+m_i}, k'_1,\dots,k'_{m_1+\dots+m_{i-1}}, k'_{m_1+\dots+m_i+1},\dots,$

$k'_m, |l'_1|, \dots, |l'_n|, l'_1, \dots, l'_n)$ with respect to the lexicographic order on $\mathbf{N}^{m+2n+p+2}$.

Similarly, $\lambda <_\sigma \lambda'$ if and only if $(ord_\sigma\lambda, ord\,\lambda, ord_1\lambda, \dots, ord_p\lambda, |l_1|, \dots, |l_n|,$ $l_1, \dots, l_n, k_1, \dots, k_m)$ is less than the corresponding $(m+2n+p+2)$-tuple for λ' with respect to the lexicographic order on $\mathbf{N}^{m+2n+p+2}$.

Two elements $\lambda_1 = \delta_1^{k_1} \dots \delta_m^{k_m} \alpha_1^{l_1} \dots \alpha_n^{l_n}$ and $\lambda_2 = \delta_1^{r_1} \dots \delta_m^{r_m} \alpha_1^{s_1} \dots \alpha_n^{s_n}$ in Λ are called *similar*, if the n-tuples $(l_1, \dots, l_n)$ and $(s_1, \dots, s_n)$ belong to the same orthant of $\mathbf{Z}^n$ (see (4.3)). In this case we write $\lambda_1 \sim \lambda_2$. We say that λ_1 *divides* λ_2 (or λ_2 is a *multiple* of λ_1) and write $\lambda_1|\lambda_2$ if $\lambda_1 \sim \lambda_2$ and there exists $\lambda \in \Lambda$ such that $\lambda \sim \lambda_1$, $\lambda \sim \lambda_2$ and $\lambda_2 = \lambda\lambda_1$.

Let K be a difference-differential field ($Char\,K = 0$) with the basic sets Δ and σ described above (and partition (3.1) of the set Δ). Let $K\{y_1, \dots, y_s\}$ be the ring of Δ-σ-polynomials over K and let ΛY denote the set of all elements λy_i ($\lambda \in \Lambda$, $1 \le i \le s$) called *terms*. Note that as a ring, $K\{y_1, \dots, y_s\} = K[\Lambda Y]$. Two terms $u = \lambda y_i$ and $v = \lambda' y_j$ are called *similar* if λ and λ' are similar; in this case we write $u \sim v$. If $u = \lambda y_i$ is a term and $\lambda' \in \Lambda$, we say that u is similar to λ' and write $u \sim \lambda'$ if $\lambda \sim \lambda'$. Furthermore, if $u, v \in \Lambda Y$, we say that *u divides v* or *v is a multiple of u*, if $u = \lambda' y_i$, $v = \lambda'' y_i$ for some y_i and $\lambda'|\lambda''$. (If $\lambda'' = \lambda\lambda'$ for some $\lambda \in \Lambda$, $\lambda \sim \lambda'$, we write $\dfrac{v}{u}$ for λ.)

Let us consider $p+1$ orders $<_1, \dots, <_p, <_\sigma$ on the set ΛY that correspond to the orders on the semigroup Λ (we use the same symbols for the orders on Λ and ΛY). These orders are defined as follows: $\lambda y_j <_i$ (or $<_\sigma$) $\lambda' y_k$ if and only if $\lambda <_i$ (respectively, $<_\sigma$)λ' in Λ or $\lambda = \lambda'$ and $j < k$ ($1 \le i \le p$, $1 \le j, k \le s$).

The order of a term $u = \lambda y_k$ and its orders with respect to the sets Δ_i ($1 \le i \le p$) and σ are defined as the corresponding orders of λ (we use the same notation $ord\,u$, $ord_i u$, and $ord_\sigma u$ for the corresponding orders).

If $A \in K\{y_1, \dots, y_s\} \setminus K$ and $1 \le k \le p$, then the highest with respect to $<_k$ term that appears in A is called the *k-leader* of A. It is denoted by $u_A^{(k)}$. The highest term of A with respect to $<_\sigma$ is called the *σ-leader* of A; it is denoted by v_A. If A is written as a polynomial in v_A, $A = I_d(v_A)^d + I_{d-1}(v_A)^{d-1} + \dots + I_0$, where all terms of $I_0, \dots, I_d$ are less than v_A with respect to $<_\sigma$, then I_d is called the *initial* of A. The partial derivative $\partial A/\partial v_A = dI_d(v_A)^{d-1} + (d-1)I_{d-1}(v_A)^{d-2} + \dots + I_1$ is called the *separant* of A. The initial and the separant of a Δ-σ-polynomial A are denoted by I_A and S_A, respectively.

If $A, B \in K\{y_1, \dots, y_s\}$, then A is said to have lower rank than B (we write $rk\,A < rk\,B$) if either $A \in K$, $B \notin K$, or $(v_A, deg_{v_A}A, ord_1 u_A^{(1)}, \dots, ord_p u_A^{(p)})$ is less than $(v_B, deg_{v_B}B, ord_1 u_B^{(1)}, \dots, ord_p u_B^{(p)})$ with respect to the lexicographic order (v_A and v_B are compared with respect to $<_\sigma$). If the vectors are equal (or $A, B \in K$) we say that A and B are of the same rank and write $rk\,A = rk\,B$.

Definition 5.1. *If $A, B \in K\{y_1, \dots, y_s\}$, then B is said to be reduced with respect to A if*

(i) *B does not contain terms λv_A such that $\lambda \sim v_A$, $\lambda_\Delta \ne 1$, and $ord_i(\lambda u_A^{(i)}) \le ord_i u_B^{(i)}$ for $i = 1, \dots, p$.*

(ii) *If B contains a term λv_A, where $\lambda \sim v_A$ and $\lambda_\Delta = 1$, then either there exists j, $1 \leq j \leq p$, such that $ord_j u_B^{(j)} < ord_j(\lambda u_A^{(j)})$ or $ord_j(\lambda u_A^{(j)}) \leq ord_j u_B^{(j)}$ for all $j = 1, \dots, p$ and $deg_{\lambda v_A} B < deg_{v_A} A$.*

If $B \in K\{y_1, \dots, y_s\}$, then B is said to be *reduced with respect to a set* $\Sigma \subseteq K\{y_1, \dots, y_s\}$ if B is reduced with respect to every element of Σ.

A set $\Sigma \subseteq K\{y_1, \dots, y_s\}$ is called *autoreduced* if $\Sigma \bigcap K = \emptyset$ and every element of Σ is reduced with respect to any other element of this set.

The proof of the following lemma can be found in [7, Chapter0, Section 17].

Lemma 5.2. *Let A be any infinite subset of $\mathbf{N}^m \times \mathbf{N}_n$ ($m, n \in \mathbf{N}$, $n \geq 1$). Then there exists an infinite sequence of elements of A, strictly increasing relative to the product order, in which every element has the same projection on $\mathbf{N}_n$.*

This lemma implies the following statement that will be used below.

Lemma 5.3. *Let S be any infinite set of terms λy_j ($\lambda \in \Lambda, 1 \leq j \leq s$) in $K\{y_1, \dots, y_s\}$. Then there exists an index j ($1 \leq j \leq s$) and an infinite sequence of terms $\lambda_1 y_j, \lambda_2 y_j, \dots, \lambda_k y_j, \dots$ such that $\lambda_k | \lambda_{k+1}$ for every $k = 1, 2, \dots$.*

Proposition 5.4. *Every autoreduced set is finite.*

Proof. Suppose that Σ is an infinite autoreduced subset of $K\{y_1, \dots, y_s\}$. Then Σ must contain an infinite set $\Sigma' \subseteq \Sigma$ such that all Δ-σ-polynomials from Σ' have different σ-leaders similar to each other. Indeed, if it is not so, then there exists an infinite set $\Sigma_1 \subseteq \Sigma$ such that all Δ-σ-polynomials from Σ_1 have the same σ-leader v. By Lemma 5.2, the infinite set $\{(ord_1 u_A^{(1)}, \dots, ord_p u_A^{(p)}) | A \in \Sigma_1\}$ contains a nondecreasing infinite sequence

$$(ord_1 u_{A_1}^{(1)}, \dots, ord_p u_{A_1}^{(p)}) \leq_P (ord_1 u_{A_2}^{(1)}, \dots, ord_p u_{A_2}^{(p)}) \leq_P \dots$$

($A_1, A_2, \dots \in \Sigma_1$ and $\leq_P$ denotes the product order on $\mathbf{N}^p$). Since the sequence $\{deg_{v_{A_i}} A_i | i = 1, 2, \dots\}$ cannot be strictly decreasing, there are two indices i and j such that $i < j$ and $deg_{v_{A_i}} A_i \leq deg_{v_{A_j}} A_j$. We see that A_j is not reduced with respect to A_i that contradicts the fact that Σ is an autoreduced set.

Thus, we can assume that all Δ-σ-polynomials of our infinite autoreduced set Σ have distinct σ-leaders similar to each other. Using Lemma 5.3, we can assume that there exists an infinite sequence $B_1, B_2, \dots$ of elements of Σ such that $v_{B_i} | v_{B_{i+1}}$ and $\left(\frac{v_{B_{i+1}}}{v_{B_i}}\right)_\Delta \neq 1$ for all $i = 1, 2, \dots$. Let $k_{ij} = ord_j v_{B_i}$ and $l_{ij} = ord_j u_{B_i}^{(j)}$ ($1 \leq j \leq p$). Obviously, $l_{ij} \geq k_{ij}$ ($i = 1, 2, \dots; j = 1, \dots, p$), so that $\{(l_{i1} - k_{i1}, \dots, l_{ip} - k_{ip}) | i = 1, 2, \dots\} \subseteq \mathbf{N}^p$. By Lemma 5.2, there exists an infinite sequence of indices $i_1 < i_2 < \dots$ such that $(l_{i_1 1} - k_{i_1 1}, \dots, l_{i_1 p} - k_{i_1 p}) \leq_P (l_{i_2 1} - k_{i_2 1}, \dots, l_{i_2 p} - k_{i_2 p}) \leq_P \dots$. Then for any $j = 1, \dots, p$, we have $ord_j\,(\frac{v_{B_{i_2}}}{v_{B_{i_1}}} u_{B_{i_1}}^{(j)}) = k_{i_2 j} - k_{i_1 j} + l_{i_1 j} \leq k_{i_2 j} + l_{i_2 j} - k_{i_2 j} = l_{i_2 j} = ord_j u_{B_{i_2}}^{(j)}$, so that B_{i_2} contains a term $\lambda v_{B_{i_1}} = v_{B_{i_2}}$ such that $\lambda_\Delta \neq 1$ and $ord_j(\lambda u_{B_{i_1}}^{(j)}) \leq ord_j u_{B_{i_2}}^{(j)}$ for $j = 1, \dots, p$. Thus, the Δ-σ-polynomial B_{i_2} is reduced with respect to B_{i_1} that contradicts the fact that Σ is an autoreduced set. $\square$

Throughout the rest of the paper, while considering autoreduced sets in the ring $K\{y_1,\dots,y_s\}$ we always assume that their elements are arranged in order of increasing rank. (Therefore, if we consider an autoreduced set of Δ-σ-polynomials $\Sigma=\{A_1,\dots,A_d\}$, then $rk\,A_1<\dots<rk\,A_d$).

Proposition 5.5. *Let $\Sigma=\{A_1,\dots,A_d\}$ be an autoreduced set in the ring $K\{y_1,\dots,y_s\}$ and let I_k and S_k $(1\le k\le d)$ denote the initial and separant of A_k, respectively. Furthermore, let $I(\Sigma)=\{X\in K\{y_1,\dots,y_s\}\,|\,X=1$ or X is a product of finitely many elements of the form $\gamma(I_k)$ and $\gamma'(S_k)$ where $\gamma,\gamma'\in\Lambda_\sigma\}$. Then for any Δ-σ-polynomial B, there exist $B_0\in K\{y_1,\dots,y_s\}$ and $J\in I(\Sigma)$ such that B_0 is reduced with respect to Σ and $JB\equiv B_0\,(mod[\Sigma])$ (that is, $JB-B_0\in[\Sigma]$).*

Proof. If B is reduced with respect to Σ, the statement is obvious (one can set $B_0=B$). Suppose that B is not reduced with respect to Σ. Let $u_i^{(j)}$ and v_i $(1\le j\le p,\ 1\le i\le d)$ be the leaders of the element A_i relative to the orders $<_j$ and $<_\sigma$, respectively. In what follows, a term w_H, that appears in a Δ-σ-polynomial $H\in R$, will be called a Σ-leader of H if w_H is the greatest (with respect to $<_\sigma$) term among all terms λv_i $(\lambda\in\Lambda,\ 1\le j\le d)$ such that $\lambda\sim v_i$, λv_i appears in H and either $\lambda_\Delta\ne 1$ and $ord_j(\lambda u_i^{(j)})\le ord_j u_H^{(j)}$ for $j=1,\dots,p$, or $\lambda_\Delta=1$, $ord_j(\lambda u_i^{(j)})\le ord_j u_H^{(j)}$ $(1\le j\le p)$, and $deg_{v_i}A_i\le deg_{\lambda v_i}H$.

Let w_B be the Σ-leader of B. Then $B=B'w_B^r+B''$ where B' does not contain w_B and $deg_{w_B}B''<r$. Let $w_B=\lambda v_i$ for some i $(1\le i\le d)$ and for some $\lambda\in\Lambda$, $\lambda\sim v_i$, such that $ord_j(\lambda u_i^{(j)})\le ord_j u_B^{(j)}$ for $j=1,\dots,p$. Without loss of generality we may assume that i corresponds to the maximum (with respect to $<_\sigma$) σ-leader v_i in the set of all σ-leaders of elements of Σ.

Suppose, first, that $\lambda_\Delta\ne 1$ (and $ord_j(\lambda u_i^{(j)})\le ord_j u_B^{(j)}$ for $j=1,\dots,p$). Then $\lambda_\Delta A_i-S_i\lambda_\Delta v_i$ has lower rank than $\lambda_\Delta v_i$, hence $T=\lambda A_i-\lambda_\sigma(S_i)\lambda v_i$ has lower rank than $\lambda v_i=w_B$. Also, $(\lambda_\sigma(S_i))^rB=(\lambda_\sigma(S_i)\lambda v_i)^rB'+(\lambda_\sigma(S_i))^rB''=(\lambda A_i-T)^rB'+(\lambda_\sigma(S_i))^rB''$. Setting $B^{(1)}=B'(-T)^r+(\lambda_\sigma(S_i))^rB''$ we obtain that $B^{(1)}\equiv B\,(mod[\Sigma])$, $B^{(1)}$ does not contain any Σ-leader, which is greater than w_B with respect to $<_\sigma$, and $deg_{w_B}B^{(1)}<r$.

Now let $\lambda_\Delta=1$, $ord_j(\lambda u_i^{(j)})\le ord_j u_B^{(j)}$ $(1\le j\le p)$, and $r_i<r$ where $r_i=deg_{v_i}A_i$. Then the Δ-σ-polynomial $(\lambda I_i)B-w_B^{r-r_i}(\lambda A_i)B'$ has all the properties of $B^{(1)}$ mentioned above. Repeating the described procedure, we arrive at a desired Δ-σ-polynomial B_0, which is reduced with respect to Σ and satisfies the condition $JB\equiv B_0\,(mod[\Sigma])$, where $J=1$ or J is a product of finitely many elements of the form $\gamma(I_k)$ and $\gamma'(S_k)$ $(\gamma,\gamma'\in\Lambda_\sigma)$. $\square$

With the notation of the last proposition, we say that the Δ-σ-polynomial B *reduces to* B_0 modulo Σ.

Definition 5.6. *Let $\Sigma=\{A_1,\dots,A_d\}$ and $\Sigma'=\{B_1,\dots,B_e\}$ be two autoreduced sets in the ring of Δ-σ-polynomials $K\{y_1,\dots,y_s\}$. An autoreduced set Σ is said to have lower rank than Σ' if one of the following two cases holds:*

(1) *There exists $k\in\mathbf{N}$ such that $k\le\min\{d,e\}$, $rk\,A_i=rk\,B_i$ for $i=1,\dots,k-1$ and $rk\,A_k<rk\,B_k$.*

(2) *$d > e$ and $rk\, A_i = rk\, B_i$ for $i = 1, \dots, e$.*

If $d = e$ and $rk\, A_i = rk\, B_i$ for $i = 1, \dots, d$, then Σ is said to have the same rank as Σ'.

Proposition 5.7. *In every nonempty family of autoreduced sets of difference-differential polynomials there exists an autoreduced set of lowest rank.*

Proof. Let Φ be a nonempty family of autoreduced sets in the ring $K\{y_1, \dots, y_s\}$. Let us inductively define an infinite descending chain of subsets of Φ as follows: $\Phi_0 = \Phi$, $\Phi_1 = \{\Sigma \in \Phi_0 | \Sigma$ contains at least one element and the first element of Σ is of lowest possible rank$\}$, ..., $\Phi_k = \{\Sigma \in \Phi_{k-1} | \Sigma$ contains at least k elements and the kth element of Σ is of lowest possible rank$\}$, ... It is clear that if A and B are any two Δ-σ-polynomials in the same set Φ_k, then $v_A = v_B$, $deg_{v_A} A = deg_{v_B} B$, and $ord_i u_A^{(i)} = ord_i u_B^{(i)}$ for $i = 1, \dots, p$. Therefore, if all sets Φ_k are nonempty, then the set $\{A_k | A_k$ is the kth element of some autoreduced set in $\Phi_k\}$ would be an infinite autoreduced set, and this would contradict Proposition 5.4. Thus, there is the smallest positive integer k such that $\Phi_k = \emptyset$. Clearly, every element of Φ_{k-1} is an autoreduced set of lowest rank in Φ. $\square$

Let J be any ideal of the ring $K\{y_1, \dots, y_s\}$. Since the set of all autoreduced subsets of J is not empty (if $A \in J$, then $\{A\}$ is an autoreduced subset of J), the last statement shows that J contains an autoreduced subset of lowest rank. Such an autoreduced set is called a *characteristic set* of the ideal J.

Proposition 5.8. *Let $\Sigma = \{A_1, \dots, A_d\}$ be a characteristic set of a Δ-σ-ideal J of the ring $R = K\{y_1, \dots, y_s\}$. Then an element $B \in R$ is reduced with respect to the set Σ if and only if $B = 0$.*

Proof. First of all, note that if $B \neq 0$ and $rk\, B < rk\, A_1$, then $rk\, \{B\} < rk\, \Sigma$ that contradicts the fact that Σ is a characteristic set of the ideal J. Let $rk\, B > rk\, A_1$ and let $A_1, \dots, A_j$ $(1 \leq j \leq d)$ be all elements of Σ whose rank is lower that the rank of B. Then $\Sigma' = \{A_1, \dots, A_j, B\}$ is an autoreduced set of lower rank than Σ, contrary to the fact that Σ is a characteristic set of J. Thus, $B = 0$. $\square$

Since for any Δ-σ-polynomial A and any $\gamma \in \Lambda_\sigma$, $ord_i(\gamma A) = ord_i A$ for $i = 1, \dots, p$, one can introduce the concept of a coherent autoreduced set of a linear Δ-σ-ideal of $K\{y_1, \dots, y_s\}$ (that is, a Δ-σ-ideal generated by a finite set of linear Δ-σ-polynomials) in the same way as it is defined in the case of difference polynomials (see [8, Section 6.5]): an autoreduced set $\Sigma = \{A_1, \dots, A_d\} \subseteq K\{y_1, \dots, y_s\}$ consisting of linear Δ-σ-polynomials is called *coherent* if it satisfies the following two conditions:

(i) λA_i reduces to zero modulo Σ for any $\lambda \in \Lambda$, $1 \leq i \leq d$.

(ii) If $v_{A_i} \sim v_{A_j}$ and $w = \lambda v_{A_i} = \lambda' v_{A_j}$, where $\lambda \sim \lambda' \sim v_{A_i} \sim v_{A_j}$, then the Δ-σ-polynomial $(\lambda' I_{A_j})(\lambda A_i) - (\lambda I_{A_i})(\lambda' A_j)$ reduces to zero modulo Σ.

The following two propositions can be proved precisely in the same way as the corresponding statements for difference polynomials, see [8, Theorem 6.5.3 and Corollary 6.5.4]).

Proposition 5.9. *Any characteristic set of a linear Δ-σ-ideal of the ring of Δ-σ-polynomials $K\{y_1, \dots, y_s\}$ is a coherent autoreduced set. Conversely, if Σ is a coherent autoreduced set in $K\{y_1, \dots, y_s\}$ consisting of linear Δ-σ-polynomials, then Σ is a characteristic set of the linear Δ-σ-ideal $[\Sigma]$.*

Proposition 5.10. *Let us consider a partial order $\preccurlyeq$ on $K\{y_1, \dots, y_s\}$ such that $A \preccurlyeq B$ if and only if $v_A | v_B$. Let A be a linear Δ-σ-polynomial in $K\{y_1, \dots, y_s\}$, $A \notin K$. Then the set of all minimal with respect to $\preccurlyeq$ elements of the set $\{\lambda A \,|\, \lambda \in \Lambda\}$ is a characteristic set of the Δ-σ-ideal $[A]$.*

Now we are ready to prove Theorem 3.1.

Proof. Let $L = K\langle \eta_1, \dots, \eta_s\rangle$ be a Δ-σ-field extension of K generated by a finite set $\eta = \{\eta_1, \dots, \eta_s\}$. Then there exists a natural Δ-σ-homomorphism Υ_η of the ring of Δ-σ-polynomials $K\{y_1, \dots, y_s\}$ onto the Δ-σ-subring $K\{\eta_1, \dots, \eta_s\}$ of L such that $\Upsilon_\eta(a) = a$ for any $a \in K$ and $\Upsilon_\eta(y_j) = \eta_j$ for $j = 1, \dots, s$. (If $A \in K\{y_1, \dots, y_s\}$, then $\Upsilon_\eta(A)$ is called the *value* of A at η; it is denoted by $A(\eta)$.) Obviously, the kernel P of the Δ-σ-homomorphism Υ_η is a prime Δ-σ-ideal of $K\{y_1, \dots, y_s\}$. This ideal is called the *defining* ideal of η over K or the defining ideal of the extension $L = K\langle \eta_1, \dots, \eta_s\rangle$. It is easy to see that if the quotient field Q of the factor ring $\bar{R} = K\{y_1, \dots, y_s\}/P$ is considered as a Δ-σ-field (where $\delta(\frac{f}{g}) = \frac{g\delta(f) - f\delta(g)}{g^2}$ and $\tau(\frac{f}{g}) = \frac{\tau(f)}{\tau(g)}$ for any $f, g \in \bar{R}$, $\delta \in \Delta$, $\tau \in \sigma^*$), then Q is naturally Δ-σ-isomorphic to the field L. The corresponding isomorphism is identity on K and maps the images of the Δ-σ-indeterminates $y_1, \dots, y_s$ in the factor ring $\bar{R}$ to the elements $\eta_1, \dots, \eta_s$, respectively.

Let $\Sigma = \{A_1, \dots, A_d\}$ be a characteristic set of the defining Δ-σ-ideal P. For any $r_1, \dots, r_{p+1} \in \mathbf{N}$, let us set $U_{r_1 \dots r_{p+1}} = \{u \in \Lambda Y | ord_i u \leq r_i$ for $i = 1, \dots, p$, $ord_\sigma u \leq r_{p+1}$, and either u is not a multiple of any v_{A_i} or for every $\lambda \in \Lambda, A \in \Sigma$ such that $u = \lambda v_A$ and $\lambda \sim v_A$, there exists $j \in \{1, \dots, p\}$ such that $ord_j(\lambda u_A^{(j)}) > r_j\}$. We are going to show that the set $\bar{U}_{r_1 \dots r_{p+1}} = \{u(\eta) | u \in U_{r_1 \dots r_{p+1}}\}$ is a transcendence basis of the field $K(\bigcup_{j=1}^{n} \Lambda(r_1, \dots, r_{p+1})\eta_j)$ over K.

Let us show first that the set $\bar{U}_{r_1 \dots r_{p+1}}$ is algebraically independent over K. Let g be a polynomial in k variables ($k \in \mathbf{N}, k \geq 1$) such that $g(u_1(\eta), \dots, u_k(\eta)) = 0$ for some $u_1, \dots, u_k \in U_{r_1 \dots r_{p+1}}$. Then the Δ-σ-polynomial $\bar{g} = g(u_1, \dots, u_k)$ is reduced with respect to Σ. (Indeed, if g contains a term $u = \lambda v_{A_i}$ with $\lambda \in \Lambda, \lambda \sim v_{A_i}$ $(1 \leq i \leq d)$, then there exists $k \in \{1, \dots, p\}$ such that $ord_k(\lambda u_{A_i}^{(k)}) > r_k \geq ord_k u_{\bar{g}}^{(k)}$). Since $\bar{g} \in P$, Proposition 5.8 implies that $\bar{g} = 0$. Thus, the set $\bar{U}_{r_1 \dots r_{p+1}}$ is algebraically independent over K.

Now, let us prove that every element $\lambda\eta_j$ $(1 \leq j \leq s, \lambda \in \Lambda(r_1, \dots, r_{p+1}))$ is algebraic over the field $K(\bar{U}_{r_1, \dots, r_{p+1}})$. Let $\lambda\eta_j \notin \bar{U}_{r_1, \dots, r_{p+1}}$ (if $\lambda\eta_j \in \bar{U}_{r_1, \dots, r_{p+1}}$, the statement is obvious). Then $\lambda y_j \notin U_{r_1, \dots, r_{p+1}}$ whence λy_j is equal to some term $\lambda' v_{A_i}$ where $\lambda' \in \Lambda$, $\lambda \sim v_{A_i}$ $(1 \leq i \leq d)$, and $ord_k(\lambda' u_{A_i}^{(k)}) \leq r_k$ for $k = 1, \dots, p$. Let us represent A_i as a polynomial in v_{A_i}: $A_i = I_0(v_{A_i})^e + I_1(v_{A_i})^{e-1} + \dots + I_e$, where $I_0, I_1, \dots I_e$ do not contain v_{A_i} (therefore, all terms

in these Δ-σ-polynomials are lower than v_{A_i} with respect to $<_\sigma$). Since $A_i \in P$,

$$A_i(\eta) = I_0(\eta)(v_{A_i})(\eta)^e + I_1(\eta)(v_{A_i})(\eta)^{e-1} + \cdots + I_e(\eta) = 0 \qquad (5.\ 1)$$

It is easy to see that the Δ-σ-polynomials I_0 and $S_{A_i} = \partial A_i/\partial v_{A_i}$ are reduced with respect to any element of the set Σ. Applying Proposition 5.8 we obtain that $I_0 \notin P$ and $S_{A_i} \notin P$ whence $I_0(\eta) \neq 0$ and $S_{A_i}(\eta) \neq 0$. Now, if we apply λ' to both sides of equation (5.1), the resulting equation will show that the element $\lambda' v_{A_i}(\eta) = \lambda\eta_j$ is algebraic over the field $K(\{\bar{\lambda}\eta_l | ord_i\bar{\lambda} \leq r_i,\ ord_\sigma\bar{\lambda} \leq r_{p+1}$, for $i = 1, \ldots, p, 1 \leq l \leq s$, and $\bar{\lambda}y_l <_1 \lambda' u^{(1)}_{A_i} = \lambda y_j\})$. Now, the induction on the set of terms ΛY ordered by $<_\sigma$ completes the proof of the fact that $\bar{U}_{r_1\ldots r_{p+1}}(\eta)$ is a transcendence basis of the field $K(\bigcup\limits_{j=1}^{s}\Lambda(r_1, \ldots, r_{p+1})\eta_j)$ over K.

Let $U^{(1)}_{r_1\ldots r_{p+1}} = \{u \in \Lambda Y | ord_i u \leq r_i$ for $i = 1, \ldots, p$, $ord_\sigma u \leq r_{p+1}$, and u is not a multiple of any v_{A_j}, $j = 1, \ldots, d\}$ and let $U^{(2)}_{r_1\ldots r_{p+1}} = \{u \in \Lambda Y | ord_i u \leq r_i$, $ord_\sigma u \leq r_{p+1}$ for $i = 1, \ldots, p$ and there exists at least one pair i, j $(1 \leq i \leq p, 1 \leq j \leq d)$ such that $u = \lambda v_{A_j}$, $\lambda \sim v_{A_j}$, and $ord_i(\lambda u^{(i)}_{A_j}) > r_i\}$. Clearly, $U_{r_1\ldots r_{p+1}} = U^{(1)}_{r_1\ldots r_{p+1}} \bigcup U^{(2)}_{r_1\ldots r_{p+1}}$ and $U^{(1)}_{r_1\ldots r_{p+1}} \bigcap U^{(2)}_{r_1\ldots r_{p+1}} = \emptyset$.

By Theorem 4.6, there exists a numerical polynomial $\phi(t_1, \ldots, t_{p+1})$ in $p+1$ variables $t_1, \ldots, t_{p+1}$ such that $\phi(r_1, \ldots, r_{p+1}) = Card\, U^{(1)}_{r_1\ldots r_{p+1}}$ for all sufficiently large $(r_1, \ldots, r_{p+1}) \in \mathbf{N}^{p+1}$, $deg_{t_i}\phi \leq m_i$ $(1 \leq i \leq p)$, and $deg_{t_{p+1}}\phi \leq n$. Furthermore, repeating the arguments of the proof of theorem 4.1 of [11], we obtain that there is a linear combination $\psi(t_1, \ldots, t_{p+1})$ of polynomials of the form (4.5) such that $\psi(r_1, \ldots, r_{p+1}) = Card\, U^{(2)}_{r_1\ldots r_{p+1}}$ for all sufficiently large $(r_1, \ldots, r_{p+1}) \in \mathbf{N}^{p+1}$. Then the polynomial $\Phi_\eta(t_1, \ldots, t_{p+1}) = \phi(t_1, \ldots, t_{p+1}) + \psi(t_1, \ldots, t_{p+1})$ satisfies conditions (i) and (ii) of Theorem 3.1.

In order to prove the last part of the theorem, suppose that $\zeta = \{\zeta_1, \ldots, \zeta_q\}$ is another system of Δ-σ-generators of L/K, that is, $L = K\langle\eta_1, \ldots, \eta_s\rangle = K\langle\zeta_1, \ldots, \zeta_q\rangle$. Let

$$\Phi_\zeta(t_1, \ldots, t_{p+1}) = \sum_{i_1=0}^{m_1} \cdots \sum_{i_p=0}^{m_p} \sum_{i_{p+1}=0}^{n} b_{i_1\ldots i_{p+1}} \binom{t_1 + i_1}{i_1} \cdots \binom{t_{p+1} + i_{p+1}}{i_{p+1}}$$

be the dimension polynomial of our Δ-σ-field extension associated with the system of generators ζ. Then there exist positive integers $h_1, \ldots, h_{p+1}$ such that $\eta_i \in K(\bigcup_{j=1}^{q}\Lambda(h_1, \ldots, h_{p+1})\zeta_j)$ and $\zeta_k \in K(\bigcup_{j=1}^{s}\Lambda(h_1, \ldots, h_{p+1})\eta_j)$ for any $i = 1, \ldots, s$ and $k = 1, \ldots, q$, whence $\Phi_\eta(r_1, \ldots, r_{p+1}) \leq \Phi_\zeta(r_1 + h_1, \ldots, r_{p+1} + h_{p+1})$ and $\Phi_\zeta(r_1, \ldots, r_{p+1}) \leq \Phi_\eta(r_1 + h_1, \ldots, r_{p+1} + h_{p+1})$ for all sufficiently large $(r_1, \ldots, r_{p+1}) \in \mathbf{N}^{p+1}$. Now the statement of the third part of Theorem 3.1 follows from the fact that for any element $(k_1, \ldots, k_{p+1}) \in E'_\eta$, the term $\binom{t_1+k_1}{k_1} \cdots \binom{t_{p+1}+k_{p+1}}{k_{p+1}}$ appears in $\Phi_\eta(t_1, \ldots, t_{p+1})$ and $\Phi_\zeta(t_1, \ldots, t_{p+1})$ with the same coefficient $a_{k_1\ldots k_{p+1}}$. The equality of the coefficients of the corresponding terms of total degree $d = deg\,\Phi_\eta$ in Φ_η and Φ_ζ can be shown in the same way as in the proof of Theorem 3.3.21 of [12]. $\square$

Example 5.11. Let us find the Δ-σ-dimension polynomial that expresses the strength of the difference-differential equation

$$\frac{\partial^2 y(x_1,x_2)}{\partial x_1^2}+\frac{\partial^2 y(x_1,x_2)}{\partial x_2^2}+y(x_1+h)+a(x)=0 \tag{5. 2}$$

over some Δ-σ-field of functions of two real variables K, where the basic set of derivations $\Delta=\{\delta_1=\frac{\partial}{\partial x_1},\delta_2=\frac{\partial}{\partial x_2}\}$ has the partition $\Delta=\{\delta_1\}\bigcup\{\delta_1\}$ and σ consists of one automorphisms $\alpha: f(x_1,x_2)\mapsto f(x_1+h,x_2)\}$ $(h\in\mathbf{R})$.

In this case, the associated Δ-σ-extension $K\langle\eta\rangle/K$ is Δ-σ-isomorphic to the field of fractions of $K\{y\}/[\alpha y+\delta_1^2 y+\delta_2^2 y+a]$ (the element $a\in K$ corresponds to the function $a(x)$). Applying Proposition 5.10 we obtain that the characteristic set of the defining ideal of the corresponding Δ-σ-extension $K\langle\eta\rangle/K$ consists of the Δ-σ-polynomials $g_1=\alpha y+\delta_1^2 y+\delta_2^2 y+a$ and $g_2=\alpha^{-1}g_1=\alpha^{-1}\delta_1^2 y+\alpha^{-1}\delta_2^2 y+y+\alpha^{-1}(a)$. With the notation of the proof of Theorem 3.1, the application of the procedure described in this proof, Theorem 4.6(iii), and formula (4.4) leads to the following expressions for the numbers of elements of the sets $U^{(1)}_{r_1r_2r_3}$ and $U^{(2)}_{r_1r_2r_3}$: $Card\,U^{(1)}_{r_1r_2r_3}=r_1r_2+2r_2r_3+r_1+r_2+2r_3+1$ and $Card\,U^{(2)}_{r_1r_2r_3}=4r_1r_3+2r_2r_3-2r_3$ for all sufficiently large $(r_1,r_2,r_3)\in\mathbf{N}^3$. Thus, the strength of equation (5.2) corresponding to the given partition of the basic set of derivations is expressed by the Δ-σ-polynomial

$$\Phi_\eta(t_1,t_2,t_3)=t_1t_2+4t_1t_3+4t_2t_3+t_1+t_2+1.$$

Example 5.12. Let K be a difference-differential (Δ-σ-) field where the basic set of derivations $\Delta=\{\delta_1,\delta_2\}$ is considered together with its partition

$$\Delta=\{\delta_1\}\bigcup\{\delta_1\} \tag{5. 3}$$

and $\sigma=\{\alpha\}$ for some automorphism α of K. Let $L=K\langle\eta\rangle$ be a Δ-σ-field extension with the defining equation

$$\delta_1^a\delta_2^b\alpha^c\eta+\delta_1^a\delta_2^b\alpha^{-c}\eta+\delta_1^a\delta_2^{b+c}\eta+\delta_1^{a+c}\delta_2^b\eta=0 \tag{5. 4}$$

where a, b, and c are positive integers. Let $\Phi_\eta(t_1,t_2,t_3)$ denote the corresponding difference-differential dimension polynomial (which expresses the strength of equation (5.4) with respect to the given partition of the set of basic derivations Δ). In order to compute Φ_η, notice, first , that the defining Δ-σ-ideal P of the extension L/K is the linear Δ-σ-ideal of $K\{y\}$ generated by the Δ-σ-polynomial

$$f=\delta_1^a\delta_2^b\alpha^c y+\delta_1^a\delta_2^b\alpha^{-c}y+\delta_1^a\delta_2^{b+c}y+\delta_1^{a+c}\delta_2^b y.$$

By Proposition 5.10, the characteristic set of the ideal P consists of f and

$$\alpha^{-1}f=\alpha^{-(c+1)}\delta_1^a\delta_2^b y+\delta_1^a\delta_2^b\alpha^{c-1}y+\delta_1^a\delta_2^{b+c}\alpha^{-1}y+\delta_1^{a+c}\delta_2^b\alpha^{-1}y.$$

The procedure described in the proof of Theorem 3.1 shows that $Card\,U^{(1)}_{r_1r_2r_3}=\phi_A(r_1,r_2,r_3)$ for all sufficiently large $(r_1,r_2,r_3)\in\mathbf{N}^3$, where $\phi_A(t_1,t_2,t_3)$ is the

dimension polynomial of the set $A = \{(a,b,c),(a,b,-(c+1))\} \subseteq \mathbf{N}^2 \times \mathbf{Z}$. Applying Theorem 4.6(iii), and formula (4.4) we obtain that $\phi_A(t_1,t_2,t_3) = 2ct_1t_2+2bt_1t_3+2at_2t_3+(b+2c-2bc)t_1+(a+2c-2ac)t_2+(2a+2b-2ab)t_3+a+b+2c-ab-2ac-2bc+2abc$. The computation of $Card\,U^{(2)}_{r_1r_2r_3}$ with the use of the method of inclusion and exclusion described in the proof of Theorem 3.1 yields the following: $Card\,U^{(2)}_{r_1r_2r_3} = (2r_3-2c+1)[c(r_2-b+1)+c(r_1-a+1)-c^2]$ for all sufficiently large $(r_1,r_2,r_3) \in \mathbf{N}^3$. Therefore, the Δ-σ-dimension polynomial of the extension L/K, which expresses the strength of equation (5.4), is as follows.

$$\begin{aligned}\Phi_\eta(t_1,t_2,t_3) &= 2ct_1t_2+2(b+c)t_1t_3+2(a+c)t_2t_3+(b+3c-2bc-c^2)t_1\\ &+(2a+2b+4c-2ab-2ac-2bc-2c^2)t_3+a+b+4c-ab-3ac-3bc\\ &+(a+3c-2ac-2c^2)t_2++2abc+2ac^2+2bc^2+2c^3-5c^2. \qquad (5.\ 5)\end{aligned}$$

The computation of the Kolchin-type univariate Δ-σ-dimension polynomial (see Theorem 2.1) via the method of Kähler differentials described in [8, Section 6.5] (by mimicking Example 6.5.6 of [8]) leads to the following result:

$$\phi_{\eta|K}(t) = \frac{D}{2}t^2 - \frac{D(D-2)}{2}t + \frac{D(D-1)(D-2)}{6} \qquad (5.\ 6)$$

where $D = a+b+c$. In this case the polynomial $\phi_{\eta|K}(t)$ carries just one invariant $a+b+c$ of the extension L/K while $\Phi_\eta(t_1,t_2,t_3)$ determines three such invariants: c, $b+c$, and $a+c$ (see Theorem 3.1(iii)), that is, Φ_η determines all three parameters a,b,c of the defining equation while $\phi_\eta(t)$ gives just the sum of these parameters.

The extension $K\langle\zeta\rangle/K$ with a Δ-σ-generator ζ, the same basic set $\Delta\bigcup\sigma$ ($\Delta = \{\delta_1,\delta_2\}$, $\sigma=\{\alpha\}$), the same partition of Δ and defining equation

$$\delta_1^{a+b}\alpha^c\zeta + \delta_2^{a+b}\alpha^{-c}\zeta = 0 \qquad (5.\ 7)$$

has the same univariate difference-dimension polynomial (5.6). However, its Δ-σ-dimension polynomial is not only different, but also has different invariants described in part (iii) of Theorem 3.1:

$$\Phi_\zeta(t_1,t_2,t_3) = 2ct_1t_2+2(a+b)t_1t_3+2(a+b)t_2t_3+At_1+Bt_2+Ct_3+E$$

where $A = B = (a+b)(1-2c)+2c$, $C = 2[1-(a+b-1)^2]$, and $E = 1+2c(a+b-1)^2$.

Two systems of algebraic difference-differential (Δ-σ-) equations with coefficients from a Δ-σ-field K are said to be *equivalent* if there is a Δ-σ-isomorphism between the Δ-σ-field extensions of K with these defining equations, which is identity on K. Our example shows that using a partition of the basic set of derivations and the computation of the corresponding multivariate Δ-σ-dimension polynomials, one can determine that two systems of Δ-σ-equations (see systems (5.4) and (5.7)) are not equivalent, even though they have the same univariate difference-dimension polynomial.

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